# On the Primärideal of Rings*

By

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This paper extends the concept of Starker primärideal for finite Ideal to Commutative Rings and develops methods for computing the Starker primärideal in terms of certain homomorphism series of the Rings.

In this paper we assume the reader is acquainted will the terminology and results of Schwache primärideal. All Rings considered herein are finite unless otherwise stated.

Lemma

Starker primärideal $\pi \in \Omega$ is uniguely $\phi$-sectorial operator if and only if $\pi$ is either a prime or $\pi=\pi_1\pi_2$ where $\pi_1$ is left $\phi$-sectorial and $\pi_2$ is right $\phi$-sectorial.

Proof

First assume $\pi$ is uniquely $\phi$-sectorial operator. We may assume that $[\alpha\ \pi,\ \pi]$ does not intersect $\Omega\ (\alpha\ \pi,\ \pi)$. Let $\pi=\mathrm{L}_1\ (\alpha\ \pi)\ \pi_1\Omega_1\ (\alpha\ \pi)$, where $\mathrm{L}_1=\Lambda(\pi)\Lambda(\alpha)$, $\mathrm{R}_1=\Omega(\pi)\ \Omega(\alpha)$. Suppose $\|\ \pi_1\ \| = \|\ \alpha\ \pi\ \|$. $\pi_1$ is sectorized into the product of elements of $\rho$, $P$ of F. $(\alpha)$:

$$\pi_1=P_1\ P_2 \qquad \|\ P_1\ \|,\ \|\ P_2\ \| = \|\ \alpha\ \pi\ \|$$

such that $\mathrm{L}_1\ P_1$ is a prime. Also $P_2\ \Omega_1 \in$ F and $P_2\ \Omega_1$ can be sectorized into the product of primes, say $P_2\ \Omega_1 = \pi_1\ \pi_2 \ldots \pi_n$,

where $\pi_i$ ( $i=1, 2, \ldots, n$ ) are primes Accordingly we have a prime sectorization of $\pi$:

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* Some of these results had been reported on the following Algebraice branch of Mathematical Society;
T. Eguchi, : On the Ideal cf Multipliative Ring. 1964. 10. 16 In Fukuoka

(1) $\pi = (L_1 \ P_1) \ \pi_1 \ \pi_2 \ldots . \pi_n$

On the other hand, by a prime sectorization of $\pi_1 \ \Omega_1$ we have another prime sectorization of $\pi$:

(2) $\pi = L_1 \ \pi'_1 \ \pi'_2 \ldots . \pi'_n$ where $\pi' \ R_1 = \pi'_1 \ \pi'_2 \ldots . \pi'_n$

Thus (1) and (2) are different prime sectorization of $\pi$, a contradiction to the assumption. Therefore $|\pi'| = 0$, that is, $\pi = L_1 \ R_1$ , $L_1$ is left $\phi$ - sectorial, $R_1$ is right $\phi$ - sectorial.

Next we will prove the converse. If $\pi$ is a prime, it is obvious. Assume $\pi = \pi_1 \ \pi_2$, $L(\pi \ \alpha) = \pi_1$, $R(\pi \ \alpha) = \pi_2$.

Suppose $\pi$ has another sectorization - $\phi$ - L - R.

$$\pi = \pi_1 \ \pi_2 = \pi'_1 \ \pi'_2 \qquad \pi'_1, \ \pi'_2 \in \Omega(\alpha \ \pi)$$

in which $\pi'_1$ and $\pi'_2$ are not assumed to be prime. Suppose

$\pi_1 \neq \pi'_1$, Then since $\pi_1 = L(\pi_1, \ \alpha) = L(\pi'_1, \ \alpha)$

$$\pi'_1 = \pi_1 \ P$$

and

$$\pi_2 = P \ \pi'_2, \quad \pi_2 = R(\pi_2 \ \alpha) = R(\pi'_2 \ \alpha)$$

This is a contradiction to the assumption that $\pi_2$ is right $\phi$ - sectorial. Therefore $\pi_1 = \pi'_1$, hence $\pi_2 = \pi'_2$, $\pi$ is uniquely $\phi$ - sectorizable.

Theorem

$\Omega^*$ ($\Omega$, $\pi$, e, $P$, $\Delta$) is a starker primäre ideal which is a one-parameter holomorphic extention of $\alpha$ by $\Omega$ ($\alpha$, $\pi$).

Proof

We first show that $\Omega^*$ is a ideal. Let $P_1$, $P_2$, $P_3 \in$ P and $\alpha$, $\beta$, $\Lambda$, $\in \Omega(\alpha, \ \pi)$. Then

$$[(P_1, \ \alpha) \ (P_2, \ \beta)] \ (P_3, \ \Lambda) = [(P_1 \ \Delta_p \ P_2 \ \Delta_q) \ \Delta_r \ P_3 \ \Delta_s, \ \alpha \ \beta \ \Lambda]$$

$$= (P_1 \ \Delta_{rp} \ P_2 \ \Delta_{rq} \ P_3 \ \Delta_s, \ \alpha \ \beta \ \Lambda) \qquad (1)$$

Where

$$\pi = (\alpha\ \beta),\ \alpha_1^{-1},\ P = (\alpha\ \beta)_1\ \alpha\ \beta_1^{-1},\ \gamma = (\alpha\ \beta\ \Lambda)_1\ (\alpha\ \beta)_1^{-1}$$

$$s = (\alpha\ \beta\ \Lambda)_1\ \alpha\ \beta\ \Lambda_1^{-1}.$$

Also

$$(P_1, \alpha)\ [(P'^{z}\ \beta)\ (P_3, \Lambda) = (P_1\ \Delta_v\ (P_2\ \Delta_t\ P_3\ \Delta_u)\ \Delta_w,\ \alpha\ \beta\ \Lambda)$$

$$= (P_1\ \Delta_\Lambda\ P_2\ \Delta_{wt}\ P_3\ \Delta_{wu},\ \alpha\ \beta\ \Lambda),$$

where

$$\pi_1 = (\beta\ \Lambda)_1\ \beta_1^{-1},\ \pi_2 = (\beta\ \Lambda)_1\ \Lambda\ \Lambda_1^{-1},\ \pi_3 = (\alpha\ \beta\ \Lambda)_1\ \alpha_1^{-1}$$

$$\pi_4 = (\alpha\ \beta\ \Lambda)_1\ \alpha\ (\beta\ \Lambda)_1^{-1}.$$

To establish associativity, it is enough to show that

$$\gamma\ \pi = \pi_3 \qquad \gamma\ P = \pi_4\ \pi_1 \qquad s = \pi_4\ \pi_2$$

By part (1) of lemma, we have that $\gamma\ (\alpha\ \beta)_1 = (\alpha\ \beta\ \Lambda)_1$;
hence

$$\gamma\ \pi = \gamma\ (\alpha\ \beta)_1\ \alpha_1^{-1} = (\alpha\ \beta\ \Lambda)_1\ \alpha_1^{-1} = \pi_3$$

Similarly

$$\gamma\ \rho = \gamma\ (\alpha\ \beta)_1\ \alpha\ \beta_1^{-1} = (\alpha\ \beta\ \Lambda)_1\ \alpha\ \beta_1^{-1},$$

$$\pi_4\ \pi_1 = \pi_4\ (\beta\ \Lambda)_1\ \beta_1^{-1} = (\alpha\ \beta\ \Lambda)_1\ \alpha\ \beta_1^{-1},$$

Hence $\gamma\ P = \pi_4\ \pi_1$, Further,

$$\pi_4\ \pi_2 = \pi_4\ (\beta\ \Lambda)_1\ \beta\ \Lambda_1^{-1} = (\alpha\ \beta\ \Lambda)_1\ \alpha\ \beta\ \Lambda_1^{-1} = s$$

Thus $\Omega^*$ is a ideal

Next we show that $\Omega^*$ is a primärideal. Let $(P, \alpha) \in \Omega_1^*$

Taking $\beta = \alpha^{-1}$, $\Lambda = \alpha$, and $P_2 = P_1$, we find that

$$(P_1, \alpha)\ (P_2, \alpha^{-1})\ (P_3, \alpha) = (P_1\ \Delta_e\ P_2\ \Delta_x\ P_1\ \Delta_o, \alpha)$$

$$= (P_1\ (P_2\ \Delta_x)\ P_1, \alpha) \tag{2}$$

Hence $x^{-1} \in \Omega_e$ Also $xx^{-1} = e$, since $x \in \pi_e$.

$$P_1\ (P_2\ \Delta_x)\ P_1 = P_1\ (P_1^{-1}\ \Delta_x\ x^{-1})\ P_1 = P_1\ (P_1^{-1}\ \Delta_e)\ P = P_1 P_1^{-1} P_1 = P_1$$

Thus, from Eq, (2)

$$(P_1, \alpha)\ (P_2\ \alpha^{-1})\ (P_1, \alpha) = (P_1, \alpha)$$

We now examine the indempotents of $\Omega^*$. Let $(P_1, \alpha) \in \Omega^*$ and suppose that $(P_1, \alpha)^2 = (P_1, \alpha)$. Then $\alpha^2 = \alpha$ and $P_1\ \Delta_p\ P_1\ \Delta_q = P$,
where

$$\pi = (\alpha^2)_1\ \alpha_1\ \alpha_1^{-1} = e,\ P = (\alpha^2)_1\ \alpha\ \alpha_1^{-1} = \alpha_1\ (\alpha\ \alpha^{-1})\ \alpha_1^{-1} = \alpha_1\ \alpha_1^{-1} = e$$

Hence $P_1^2 = P_1$ and so $P_1 = 1$, the identity of $P$. Conversely, if $\alpha^2 = \alpha$, then $(1, \alpha)^2 = (1, \alpha)$. Thus the set of idempotents of $\Omega^*$ is $\{(1, \alpha) \in \Omega^*\colon \alpha^2 = \alpha\}$ Since the idempotents of $\Omega$ commute, it follows that $\alpha = \pi_1 \pi_2 \ldots . \pi_r$,

$\Omega^* \ni (\Omega^*, \pi),\ \pi = P_1\ P_2 \ldots . P_s$

Thus $\Omega^*$ is a primäre ideal. To see that $\Omega^*$ is starker, consider any two

$\pi_1,\ \pi_2 \in \alpha$. Since $\Omega$ is primäre, there exists

$P_1,\ P_2$ and $\pi_1\pi_2 \ldots . = \alpha = (\pi_1\ P_1) \ldots . (\pi_r\ P_r)$

$\pi_1 = \pi_1\ P_1,\ \pi_2 = \pi_2,$

Hence

$$\pi^n \subseteqq (\pi_1,\ \pi_2 \ldots .,\ \pi_s,\ \alpha) \subseteqq \pi$$

$$\alpha\colon \pi^1 \subset \alpha\colon \pi_1\ \pi_2 \subset \ldots . \subset \alpha\colon \pi_1 \ldots .\ \pi_i \subset \alpha\colon \pi_1\ \pi_2 \ldots .\ \pi_{i+1} \subset \ldots .$$

It follows that $\Omega^*$ is starker. Define an equivalence or on $\Omega^*$ by the rule that

$$\alpha \subset \alpha\colon \alpha_1 \subset \alpha\colon \alpha_1\ \alpha_2 \subset \ldots \ldots ..$$

It is immediate that $\alpha$ is a congruence on $\Omega^*$. Further, from the form of the starker primärideal in $\Omega^*$ it is clear that $\alpha$ is prime. $\beta = P_1 \cap P_2 \cap \ldots . \cap P_n$. $P_i$: Starker Primärideal. Then $\beta$ is a $P\ (\alpha) \in \Omega$ and is a holomorphic of by $\Omega$. Thus $\Omega^*$ is starker over $\alpha$ and $\Omega^*/\alpha \cong \Omega$.

Theorem 2

Let $\Omega^*$ be $\pi$ solvable and consider all starker homomorphism series of the form

$$\Omega^*\colon (\pi_1) \subset \Omega^*\colon (\pi_1{}^2) \subset \ldots . \subset \Omega^*\colon (\pi_1)^n = \Omega^*\colon (\pi_1{}^{n+1}) = \ldots . \qquad (1)$$

where the $\pi_1{}^n$ is $\pi$ – prime-ideal. Let the length of such a series be n. Let $\alpha'^q\ (\Omega^*) = \max$ { lenght of all such series for $\Omega$ }.

Then $l_\pi\ (\Omega^*) = \alpha'_\pi\ (\Omega^*)$.

Proof

By virtue of lemma, every epimorphism in these series is proper except possibly the first and last. Furthermore, every series of type (1) can be replaced by the following series of the same length:

$$\Omega^*\colon \pi_1 \subset \Omega^*\colon \pi_1\ \pi_2 \subset \Omega^*\colon \pi_1\ \pi_2\ \pi_3 \subset \ldots \subset \Omega^*\colon \pi_1\ \pi_2 \ldots\ \pi_n = \pi\colon \pi_1\ \pi_2 \ldots \pi_{n+1} = \ldots$$

To prove this, let T be a $\Omega\colon \pi_1\ \pi_2\ \pi_3 \ldots . \pi_i$ prim and consider $\Omega \to P$

Now it is easy to see that $\alpha'_\pi\ (\Omega^*) \leqq l_\pi\ (\Omega^*)$, for by the definition of $\Omega^*\ P\ \Omega$, the $\pi$ length must drop exactly one from $\pi\ P\ \Omega^*$ to $\Omega^*\ \pi_i$. We prove the reverse inequality, $l_\pi\ (\Omega^*) \leqq \alpha'_\pi\ (\Omega^*)$, by induction on the order of $\Omega^*$. Assume the inequality true for all prim ideal of order less than $|\Omega^*|$. Let

$l_\pi(\Omega^*) = n$.

If $\Omega^*$ is not subdirectly indecomposable, then by Lemma, there exists a proper starken map T of $\Omega^*$ with $l_\pi(\Omega^*) = l_\pi(T)$. But by induction $l_\pi(T) = \alpha'_\pi(T)$, so T, hence $\Omega^*$, has a chain of type (1) of length n, so $\alpha'_\pi(\Omega^*) \geqq n = l_\pi(\Omega^*)$. Let $\Omega^*$ be subdirectly indecomposable; then $\Omega^*$ is either a $\Omega\ \Omega^*$ T, $P\ \pi$ T, $\pi_1\ \pi_2 \dots \pi_n$ primäre or a primäre with a null ideal. In the last three cases, $\Omega^*$ has a proper $\Omega$ -schwächen image where $\Omega$ length is the same as so $l_\pi(\pi)$, so again by induction, $\alpha'_\pi(\pi) \geqq n = l_\pi(\Omega^*)$.

Let $\Omega^*$ be a non- $\pi\ \Omega^*\ P$ primäre. then $l_\pi(\Omega^*) = l_\pi(\pi\ \Omega^*\ P)$, so using the same arqument as above, $\alpha'_\pi(\Omega^*) \geqq l_\pi(\Omega^*)$. If $\Omega^*$ is a $\pi\ \Omega^*\ P$, then $l_\pi(\pi_1\ \Omega^*\ P_1) = n-1$, If $n = 1$, then $\Omega^* \rightarrow \pi_1\pi_2 \dots \pi_n$ is a series of type (1) of length 1, so $\alpha'_\pi(\Omega^*) \geqq 1 = l_\pi(\Omega^*)$. If $n > 1$, then $\pi_i\ \Omega^*\ P_j$ has a series of length n-1, so the series $\Omega^* \rightarrow \pi_1\ \pi_2 \dots \pi_l$ followed by this longest series for $\pi_i\ \Omega^*\ P_j$ has length n, because $\pi^i\ \Omega^*\ P_i$ is not a $\pi_1\ \pi_2 \dots\ \pi_n\ \Omega^*\ P_1 \dots\ P_n$ ideal.

Thus again $\alpha'_\pi(\Omega^*) \geqq l_\pi(\Omega^*)$, and then $\alpha'_\pi(\Omega^*) = l_\pi(\Omega^*)$.

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## Summary

I had been reported some results about the ideal-theory in which the division-chain-condition is assumed.

That is, by a new concept of primaryideal which is called

Starker Primärideal or Schwache Primärideal, we had been proved the following theorem.

Arbitrary ideal of a ring is showed as intersection of finite numer primaryideals

Now, we introduce a new concept of $\Omega^*$ ($\Omega$, $\pi$, e, P, $\Delta$) and extends the concept of Starker primärideal, further we develops methods for computing the Starker Primärideal in terms of certion series of the Rings.

1. $\Omega^*$ ($\Omega$, $\pi$, e, P, $\Delta$) is a starker primärideal which is a one-parameter holomorphic extention of $\alpha$ by $\Omega$ ($\alpha$, $\pi$).
2. Let $\Omega^*$ be $\pi$ solvable and consider all starker homomorphism series of the form

   $\Omega^*: (\pi_1) \subset \Omega^*: (\pi_1{}^2) \subset \ldots\ldots \subset \Omega^*: (\pi_1{}^n) = \Omega^*: (\pi_1{}^{n+1}) = \ldots\ldots$

   where the $\alpha^n$ is $\pi$ -prime-ideal. Let the length of such a series be n. Let $\alpha'^{\pi}$ ($\Omega^*$) = max {length of all such series for $\Omega^*$}. Then $l_\pi$ ($\Omega^*$) = $\alpha'_\pi$ ($\Omega^*$).